Gauss’ Theorem of Egregium and its Applications to Maps and Surfaces

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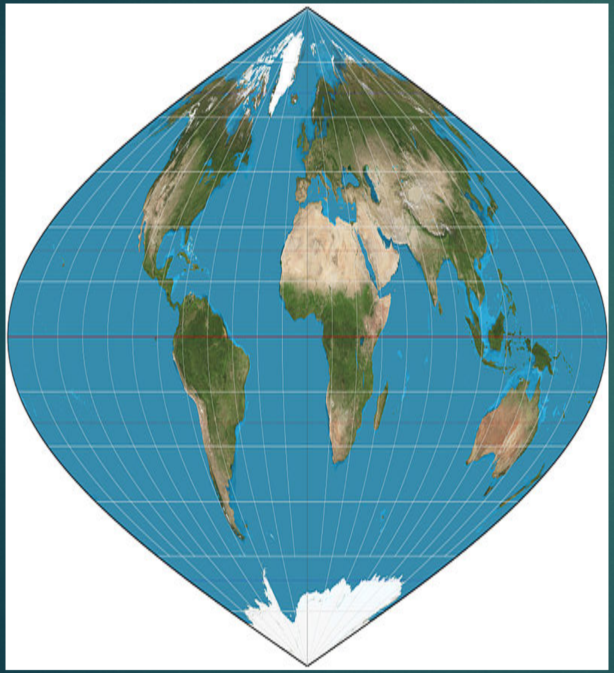
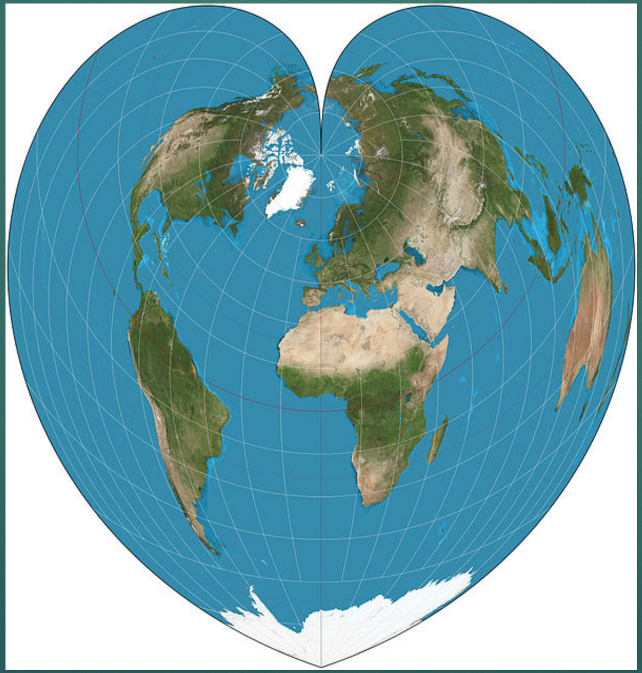
Map Projections

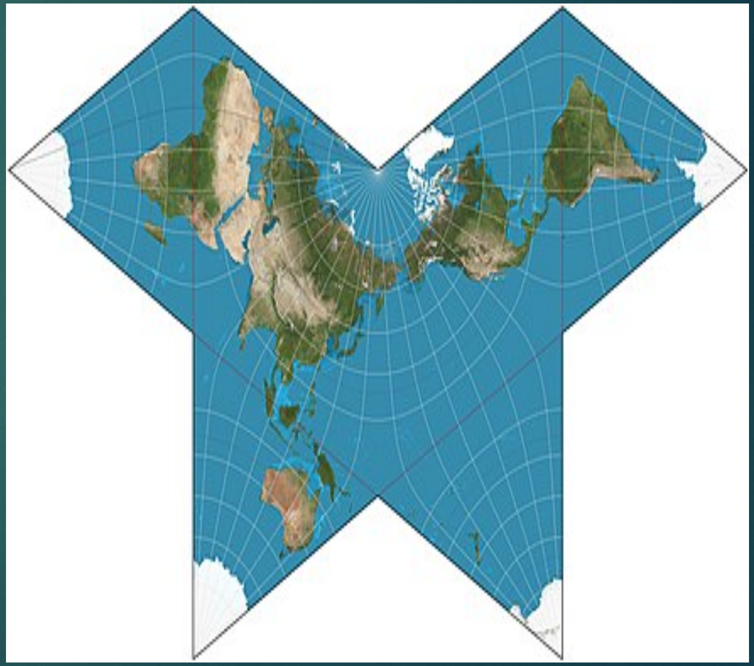
Sinusoidal Projection
Equal-area maps
preserve area measure

Werner Projection
An equidistant projection preserves distances from one or two special points to all other points
In this case, Distances from the North Pole are preserved, in equatorial aspect

Butterfly Map
Compromise Projection





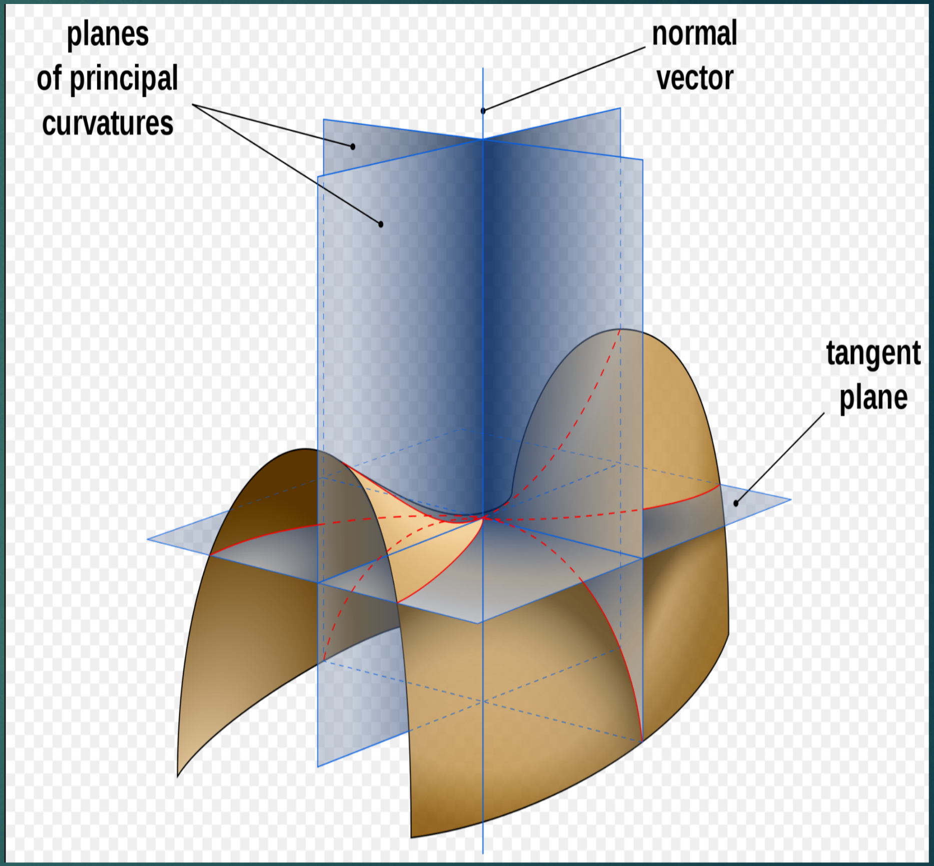


Definitions and Terminology

- The idea of <u>curvature</u> is that we can assign an amount/value to how much a curve deviates from being a straight line, or a surface deviates from being a plane.
- We will try to understand the concept of Principal Curvatures and Gaussian Curvatures:

Formal Definition : Two principal curvatures (k_1 and k_2) at given point of a surface are the eigenvalues of the shape operator at the point.

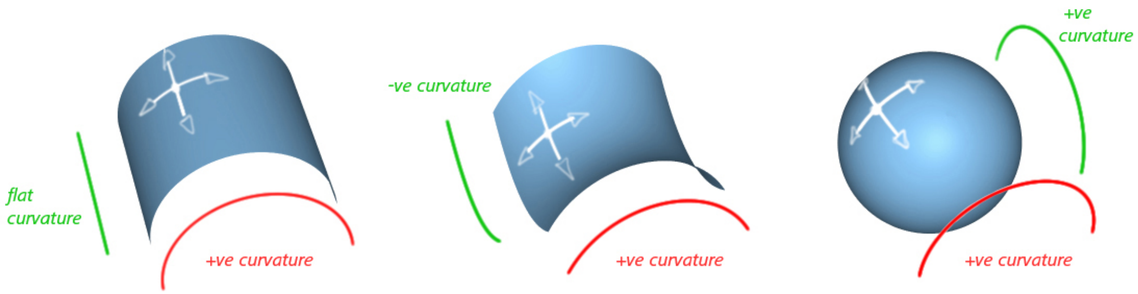
Main Idea : They measure how the surface bends by different amounts in different directions at that point.



Classification

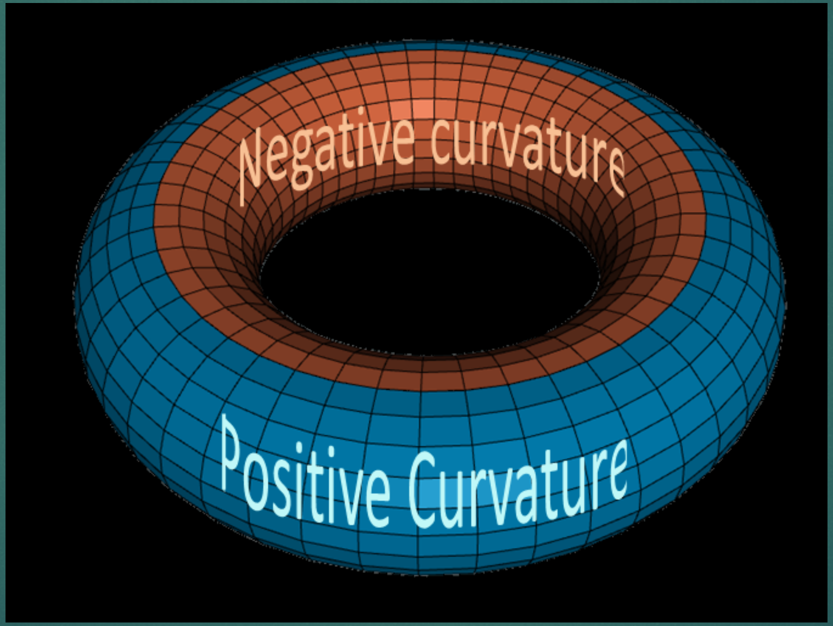
Surface point classes[1]

k_1		> 0	= 0	< 0
k_2	> 0	Concave ellipsoid	Concave cylinder	Hyperboloid surface
	= 0	Concave cylinder	Plane	Convex cylinder
	< 0	Hyperboloid surface	Convex cylinder	Convex ellipsoid



The Gaussian curvature or Gauss curvature K of a surface at a point is then defined as the product of the principal curvatures at the given point.

$$K = k_1 \; k_2$$



Gauss' Theorema Egregium

- (Latin Translated) If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged, or
- The Gaussian curvature is determined by only the first fundamental form, or
- The Gaussian curvature of an embedded smooth surface in $\mathbf{R}^3$ is invariant under the local isometries.

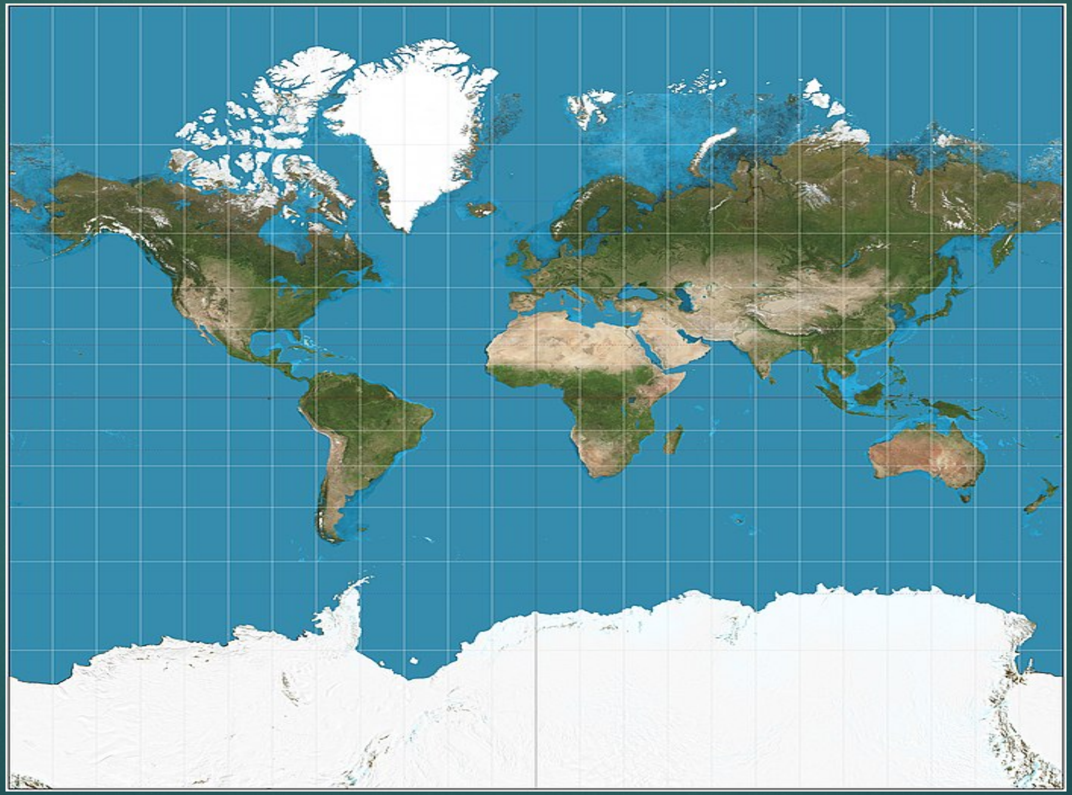
Significant Takeaways :

- Gaussian curvature can be determined entirely by measuring angles, distances and their rates on a surface, without reference to the way the surface is embedded.
- The Gaussian curvature of a surface does not change if one bends the surface without stretching it.
- The Gaussian curvature is an **intrinsic invariant** of a surface.

Interesting Applications

1. Map Making

- There are a lot of different maps of the Earth. The most famous and standard is the Mercator projection



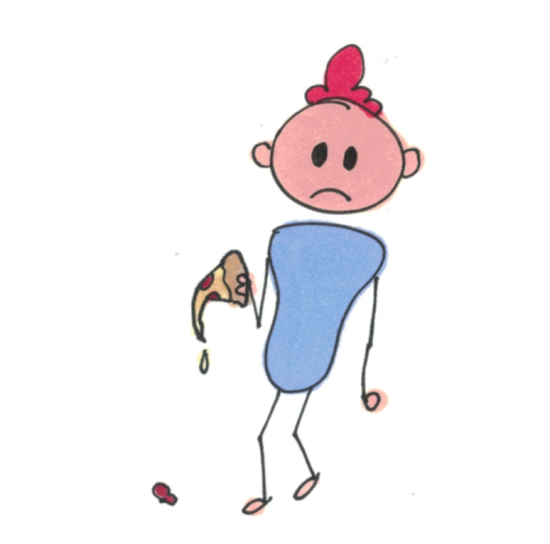
- Maps are printed or displayed on flat surfaces, and flat surfaces have zero Gaussian curvature. Spheres, like the Earth, have positive Gaussian curvature. But remember that Gaussian curvature can be measured using only distances and angles and areas.

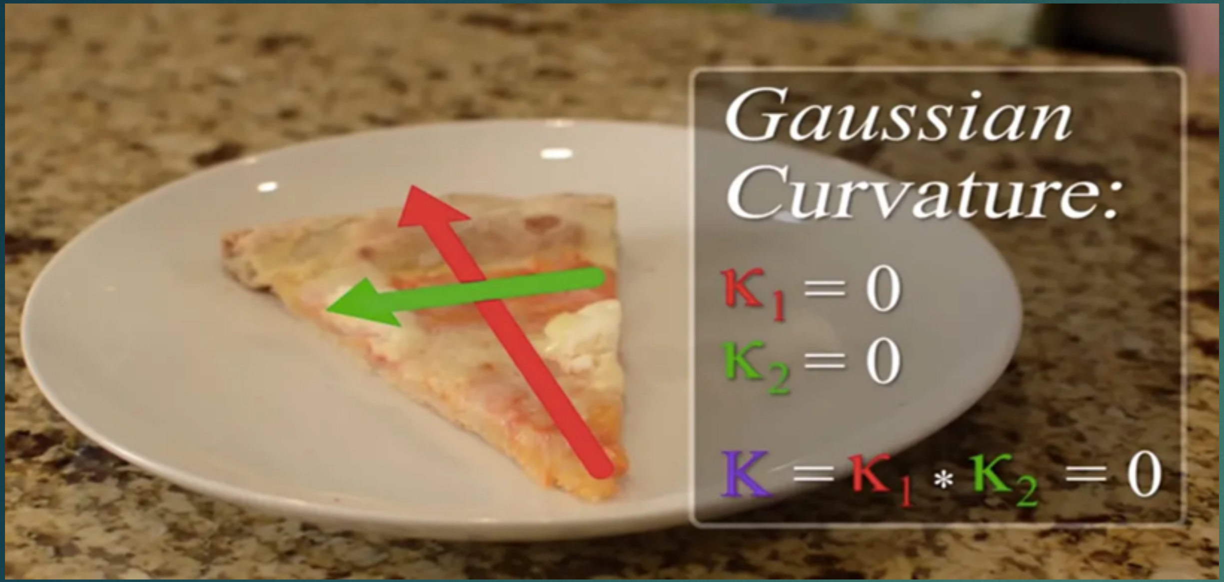
- Since a map and a sphere have different curvatures, it means that something must distort. So if you want angles to be correct, your distances and areas will be wrong. If you want areas to be correct, your angles will be off.

- Gauss’s theorem shows that no flat map can represent the Earth perfectly.

2. Eating Pizza

- Some pizza comes with a very thin, floppy crust. If you try to just pick it up, it’s going to flop over and make a mess of everything.




Gaussian Curvature:
$\kappa_1 = 0$
$\kappa_2 = 0$
$K = \kappa_1 * \kappa_2 = 0$


ZERO CURVATURE MAINTAINED
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ZERO CURVATURE MAINTAINED
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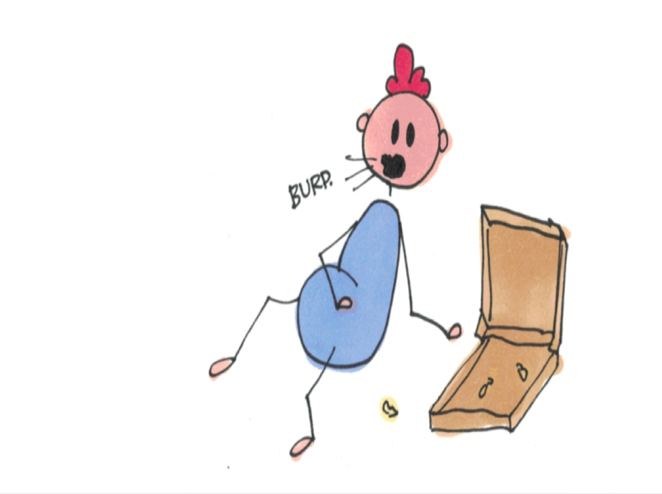
To Summarize

- Principal Curvatures
- Gaussian Curvatures
- Gauss' Theorem of Egregium
- Map Making
- Eating Pizza



Sources Cited

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Thank You